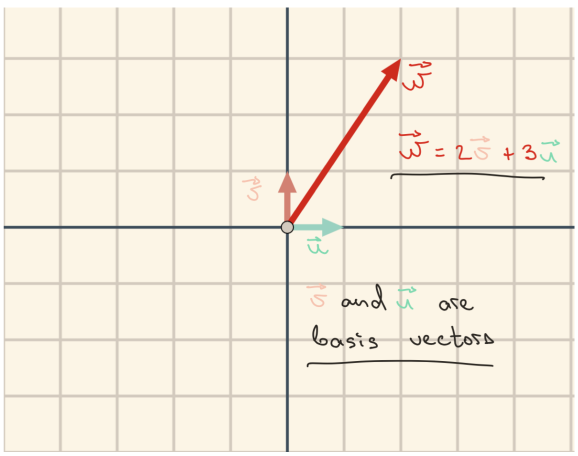
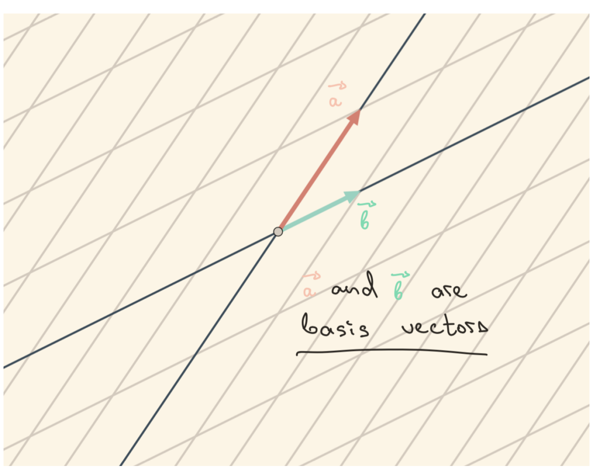
Matrix Representation of Linear Transformation

So far we have referred to vectors as simply arrows in space but I would like to start this section by talking about association between arrows in space and pairs of numbers.

Example: Imagine a 2 dimensional plane passing through the origin. Now imagine 2 orthogonal unit basis vectors **u** and **v** starting at the origin. Voila, you have just constructed the very famous Cartesian Coordinate system! Other vectors that lie in that plane can be represented as a linear combination of **u** and **v** as you can see here:



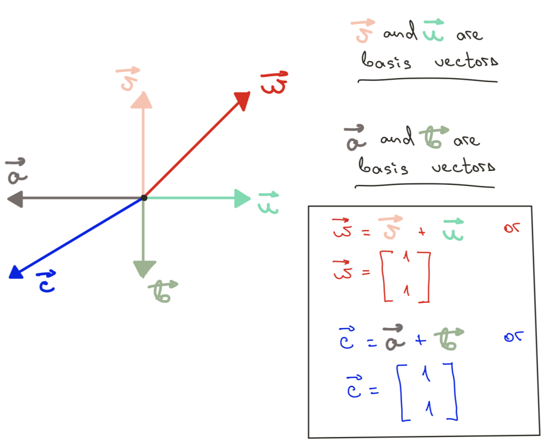
u and **v** is the basis of the plane. But they are not the only basis! Consider the following 2 vectors called **a** and **b**:



Since every vector in the plane can also be obtained by a linear combination of **a** and **b** then **a** and **b** is also a basis! Moreover, any 2 vectors that don’t lie on the same line form a basis of the plane. As you can see, there are infinitely many ways to select the basis. But don’t think that the Cartesian basis that was shown above is the best basis. In the MathTheBeautiful YouTube channel Pavel Grinfeld stated that all bases are created equal and that is the particular problem that should dictate the choice of basis.

When the basis is chosen, any other vector is represented as a linear combination of that basis. As a result, we can start associating vectors that are arrows in space to the pairs of numbers. For example, if **v** and **u** are basis vectors and if **w** = 2**v** + 3**u** then **w** can be represented as the following pair of numbers: [2, 3].

Important: Again, when the basis is chosen, any other vector is represented in terms of that basis. Pairs of numbers association only makes sense when the basis is known. If the basis is not chosen then pairs of numbers don’t make any sense.



In the figure above both **w** and **c** can be associated with the same pair of numbers, namely [1, 1]. But since **w** and **c** are represented with respect to different bases, they are completely different vectors.

Now let's put the concept of linear transformation and basis together. Once again, imagine a 2 dimensional plane passing through the origin. Let **u** and **v** be the basis vectors of that plane. Let's take a vector **w** in the plane which is an arbitrary linear combination of **u** and **v** (for example, **w** = 2**u** + 3**v**, thus you can associate **w** to the following pair of numbers [2, 3]). Now let's take an arbitrary linear transformation T (you can treat T to be the reflection which was discussed in the Linear Transformation section if you feel comfortable about it). I propose you to apply linear transformation T to the vector **w**:

$T(\mathbf{w}) = T(2\mathbf{u} + 3\mathbf{v}) = 2\,T(\mathbf{u}) + 3\,T(\mathbf{v})$

Where the first equality holds because **w** is represented as the following linear combination of **u** and **v**: **w** = 2**u** + 3**v**. And the second equality holds because of the properties of linear transformation discussed in the Linear Transformation section.

An important observation to make here is that the resulting vector T(**w**) can be represented as a linear combination of T(**u**) and T(**v**) - transformed basis vectors. Moreover, the scalars remain the same (2 and 3). So if we know where the basis vectors go under the linear transformation, we know everything about linear transformation!

We know that under linear transformation T vector **u** becomes vector T(**u**). Vector T(**u**) is a vector in the same plane as **u** and **v** and thus it can be uniquely written as a linear combination of **u** and **v**: T(**u**) = a**u** + b**v**, where a and b are some real numbers.

As a result, you can associate T(**u**) with a pair of numbers: [a, b]. Similarly, we know that under linear transformation T vector **v** becomes vector T(**v**). Analogically, vector T(**v**) is a vector in the same plane as **u** and **v** and thus it can be uniquely written as a linear combination of **u** and **v**: T(**u**) = c**u** + d**v**, where c and d are some real numbers. So you can associate T(**v**) with a pair of numbers: [c, d]. Once again, to know everything about the linear transformation you only need to know where basis vectors go. And for this you should only know 2 pairs of numbers - [a, b] and [c, d] as mentioned above! Incredible!

Putting everything together:

1) **u** and **v** are the basis vectors of some 2 dimensional vector space (plane);

2) **w** = 2**u** + 3**v**. Or **w** = [2; 3].

3) T(**w**) = T(2**u** + 3**v**) = 2T(**u**) + 3T(**v**) = 2 [a, b] + 3 [c, d].

Now I propose you review matrix-vector multiplication. There are several ways to multiply a matrix and a vector:

1) $$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = 2\begin{bmatrix} a \\ b \end{bmatrix} + 3\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 2a+3c \\ 2b+3d \end{bmatrix}$$

2) $$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2a+3c \\ 2b+3d \end{bmatrix}$$

Did you notice that the intermediate result of the first way is exactly the same as the expression for T(**w**), namely 2 [a, b] + 3 [c, d]? All you need to do is construct a matrix where the first column is [a, b] and the second column is [c, d]!

Result:

Let **u** and **v** be the basis vectors of 2 dimensional vector space (plane). Assume that under some linear transformation T vector **u** goes to vector T(**u**) such that T(**u**) = a**u** + b**v** and vector **v** goes to vector T(**v**) such that T(**v**) = c**u** + d**v**. Then the linear transformation T can be represented as a 2x2 matrix A such that [a, b] is the first column and [c, d] is the second column. Moreover, T(**w**) = A**w**.

Note:

The above can be applied to the vector space of any dimension!

Now we are ready to answer the question that was posted long ago in the tutorial devoted to Linear Transformations. How many different linear transformations except reflection are there? Well, first we know that linear transformation does not make sense if the basis is not chosen. So let's first choose a basis. Now, let's decide where the basis vectors would go under the linear transformation. Once we know this we can go ahead and create our linear transformation matrix. And congratulations, you have just created your own linear transformation! I think that I convinced you that there are infinitely many linear transformations because there are infinitely many locations where basis vectors can go under linear transformation.